Duality symmetry, strong-coupling expansion and universal critical amplitudes in two-dimensional Φ^4 field models

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Abstract. We show that the exact beta-function $\beta(g)$ in the continuous two-dimensional $g\Phi^4$ model possesses the Kramers–Wannier duality symmetry. The duality symmetry transformation $\tilde{g} = d(g)$ such that $\beta(d(g)) = d'(g)\beta(g)$ is constructed and the approximate values of g^* computed from the duality equation $d(g^*) = g^*$ are shown to agree with the available numerical results. The calculation of the beta-function $\beta(g)$ for the 2D scalar $g\Phi^4$ field theory based on the strong-coupling expansion is developed and the expansion of $\beta(g)$ in powers of g^{-1} is obtained up to order g^{-8}. The numerical values calculated for the renormalized coupling constant g_+^* are in reasonably good agreement with the best modern estimates recently obtained from the high-temperature series expansion and with those known from the perturbative four-loop renormalization-group calculations. The application of Cardy's theorem for calculating the renormalized isothermal coupling constant g_c of the 2D Ising model and the related universal critical amplitudes is also discussed.

1. Introduction

In this paper we study mainly the symmetry properties of the beta-function $\beta(g)$ for the two-dimensional (2D) $g\Phi^4$ theory, regarded as a continuum limit of the exactly solvable 2D Ising model. In contrast to the latter, the 2D $g\Phi^4$ theory is not an integrable quantum field theory. This means, in particular, that the theory does not possess the factorized scattering matrix, and therefore that the thermodynamic Bethe ansatz method cannot be applied at all.

Thus, despite the fact that the 2D Ising model at $h = 0$ can be solved by many different methods (see [1] for an excellent review), the beta-function $\beta(g)$ of its continuum limit is to date known only in the four-loop approximation within the framework of conventional perturbation theory at fixed dimension $d = 2$ [2–4]. Calculations of beta-functions are of great interest in statistical mechanics and quantum field theory. The beta-function contains the essential information on the renormalized coupling constant g^*, this being important for constructing the equation of state of the 2D Ising model, for example, which still remains a challenging problem, rich in applications. This and other considerations do not allow us to regard the 2D Ising model as having been fully solved.

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The 2D Ising model and some other lattice spin models are known to possess the remarkable Kramers–Wannier (KW) duality symmetry, which plays an important role both in statistical mechanics and in quantum field theory [5–7]. The self-duality of the isotropic 2D Ising model means that there exists an exact mapping between the high-T and low-T expansions of the partition function [7]. In the transfer-matrix language this implies that the transfer matrix of the model under discussion is covariant under the duality transformation. If we assume that the critical point is unique, the KW self-duality would yield the exact Curie temperature of the model. This holds for a large set of lattice spin models including systems with quenched disorder (for a review see [7, 8]).

Over 20 years ago the KW self-duality was shown to be equivalent to a Fourier transformation in target space [9]. Also, it was recognized long ago that self-duality combined with some special algebraic properties of a model leads to the existence of an infinite set of conserved charges [10]. Duality is thus known to impose some important constraints on the exact beta-function [11, 12].

The other main purpose of this paper is to develop a strong-coupling expansion for the calculation of the beta-function of the 2D scalar $g\Phi^4$ theory as an alternative approach to standard perturbation theory. It will then be of interest to match this expansion with the results of a four-loop approximation (where possible) by constructing a smooth interpolation with respect to g. It is in fact well known from quantum field theory and statistical mechanics that any strong-coupling expansion is closely connected with a suitable high-temperature (HT) series expansion for a lattice model [1, 7]. From the field-theoretical point of view the HT series are nothing but strong-coupling expansions for field models, the lattice being considered as a technical device to define cutoff-regularized field theories.

Recently, the high-temperature series expansions and perturbative calculations for the $g\Phi^4$ field theory at fixed dimensions $d < 4$ have been a topic of intense study (for references see below). Computing critical exponents and various critical amplitude ratios from series expansion data has a long history going back to the early 1960s. Nowadays there are a good number of papers containing a large body of information for the N-vector model defined on different lattices for $d = 2, 3, 4, 5$ and arbitrary N [13–15]. It is remarkable that the HT series data for the zero-field susceptibility χ and the second correlation moment μ_2 of the N-component classical Heisenberg ferromagnet have been extended up to the order $K^{21} (K = J/T)$, the data for the second field derivative of the susceptibility (χ_4) being available through to order K^{17}. Having been equipped with this information, one may try to employ different techniques of resummation of the existing HT series expansions, like Padé approximants or more subtle approaches, for computing critical exponents and universal critical amplitude ratios [13, 16–21]. It is worth noting that the strong-coupling behaviour of the $g\Phi^4$ theory has recently been treated within the framework of a variational perturbative approach [22].

This paper is organized as follows. In section 2 we set up basic notation and define the duality symmetry transformation $\tilde{g} = d(g)$. Then it is proved that $\beta(d(g)) = d'(g)\beta(g)$. An approximate expression for $d(g)$ providing good estimates for g_+^* (the renormalized fixed-point coupling constant along the isochore line) is found. In section 3 the HT series expansion data are used to obtain the strong-coupling expansion of $\beta(g)$ for the 2D O(N)-symmetric $g\Phi^4$ theory in powers of $1/g$ up to the order of g^{-8}. Some numerical estimates for the renormalized coupling constant g_+^* above T_c are obtained. We then compare the fixed-point values found to those already known from the four-loop renormalization-group (RG) calculations and from the HT series expansions. In section 4 we also discuss the application of Cardy's formula both for the exact calculation of the renormalized isothermal coupling constant g_c^* at T_c and, for some universal critical amplitudes, along the isothermal critical line. Finally section 5 contains

some concluding remarks. The appendix presents a simple derivation of the correlation length ξ and of the exact beta-function $\beta_{\text{Ising}}(T)$ for the lattice 2D Ising model, where the temperature T plays the role of an effective coupling constant, and we discuss some of their properties.

2. Duality symmetry of the beta-function

We begin by considering the classical Hamiltonian of the 2D Ising model (in the absence of an external magnetic field), defined on a square lattice with periodic boundary conditions; as usual:

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \tag{2.1}$$

where $\langle i, j \rangle$ indicates that the summation is over all nearest-neighbouring sites; $\sigma_i = \pm 1$ are spin variables and J is a spin coupling. The standard definition of the spin-pair correlation function reads

$$G(R) = \langle \sigma_R \sigma_0 \rangle \tag{2.2}$$

where $\langle \cdots \rangle$ denotes a thermal average.

The correlation length may be defined in many different ways, all definitions being equivalent to each other in the close vicinity of the critical point [13]. This, in fact, reflects the arbitrariness somewhat inherent in any renormalization scheme. The statistical mechanics definition of the correlation length is given by [23]

$$\xi^2 = \left. \frac{\mathrm{d} \ln G(p)}{\mathrm{d}p^2} \right|_{p=0}. \tag{2.3}$$

The quantity ξ^2 is known to be conveniently expressed in terms of the spherical moments of the spin correlation function itself, namely

$$\mu_l = \sum_{\boldsymbol{R}} (R/a)^l G(\boldsymbol{R}) \tag{2.4}$$

with a being some lattice spacing. It is easy to see that

$$\xi^2 = \frac{\mu_2}{2d\mu_0} \tag{2.5}$$

where d is the spatial dimension (in our case $d = 2$). It should be mentioned that the above definition of ξ differs from the one used in other related approaches, e.g. [11].

The 2D Ising model near T_c is known to be equivalent to the $g\Phi^4$ theory with a one-component real order parameter. In order to extend the KW duality symmetry to the continuous field theory we have need for a 'lattice' model definition of the coupling constant g, equivalent to the conventional one exploited in the RG approach. The renormalization coupling constant g of the $g\Phi^4$ theory is closely related to the fourth derivative of the 'Helmholtz free energy', namely $\partial^4 F(T, m)/\partial m^4$, with respect to the order parameter $m = \langle \Phi \rangle$. It may be defined as follows (see [13, 14, 24] and references therein):

$$g(T, h) = -\frac{(\partial^2 \chi / \partial h^2)}{\chi^2 \xi^d} + 3 \frac{(\partial \chi / \partial h)^2}{\chi^3 \xi^d} \tag{2.6}$$

where χ is the homogeneous magnetic susceptibility

$$\chi = \int d^2x \, G(x). \tag{2.7}$$

It is in fact easy to show that $g(T, h)$ in equation (2.6) is merely the standard four-spin correlation function taken at zero external momenta. The renormalized coupling constant of the critical theory is defined by the double limit

$$g^* = \lim_{h \to 0} \lim_{T \to T_c} g(T, h) \tag{2.8}$$

and it is well known that these limits do not commute with each other. As a result, g^* is a path-dependent quantity in the thermodynamic (T, h) plane [13].

Here we are mainly concerned with the coupling constant on the isochore line $g(T > T_c, h = 0)$ in the disordered phase and with its critical value

$$g_+^* = \lim_{T \to T_c^+} g(T, h = 0) = -\frac{\partial^2 \chi / \partial h^2}{\chi^2 \xi^d}\bigg|_{h=0}. \tag{2.9}$$

The couplings $g_\pm^*$ are of great interest for calculating the equation of state. Notice that the two different fixed-points values g_+^* (defined by the above equation (2.9)) and g_-^* (defined by the analogous limit procedure for $T \to T_c^-$) computed above and below the Curie temperature differ vastly from each other owing to the broken symmetry of the ordered phase.

The 'lattice' coupling constant g_+^* defined in equation (2.9) is in a given correspondence with the temperature T_c. We shall see that it will be more convenient to deal with a new variable $s = \exp(2K)\tanh(K)$, where $K = J/T$. The standard KW duality transformation is known to be as follows [6, 7]:

$$\sinh(2\tilde{K}) = \frac{1}{\sinh(2K)}. \tag{2.10}$$

It follows from the definition that s transforms as $\tilde{s} = 1/s$; this implies that the correlation length of the 2D Ising model (see also the appendix) $\xi^2 = s/(1-s)^2$ is a self-dual quantity. Now, on the one hand, we have the formal relation

$$\xi \frac{ds(g)}{d\xi} = \frac{ds(g)}{dg}\beta(g) \tag{2.11}$$

where $s(g)$ is defined as the inverse function of $g(s)$, i.e. $g(s(g)) = g$ and the beta-function is given, as usual, by

$$\xi \frac{dg}{d\xi} = \beta(g). \tag{2.12}$$

On the other hand, it is shown in the appendix that

$$\xi \frac{ds}{d\xi} = \frac{2s(1-s)}{(1+s)}. \tag{2.13}$$

From equations (2.11)–(2.13), a useful representation of the beta-function in terms of the $s(g)$ function thus follows:

$$\beta(g) = \frac{2s(g)(1-s(g))}{(1+s(g))(ds(g)/dg)}. \tag{2.14}$$

Let us define the dual coupling constant $\tilde{g}$ and the duality transformation function $d(g)$ as

$$s(\tilde{g}) = \frac{1}{s(g)} \qquad \tilde{g} \equiv d(g) = s^{-1}\left(\frac{1}{s(g)}\right) \tag{2.15}$$

where $s^{-1}(x)$ stands for the inverse function of $x = s(g)$. It is easy to check that a further application of the duality map $d(g)$ gives back the original coupling constant, i.e. $d(d(g)) = g$, as it should be. Notice also that the definition of the duality transformation given by equation (2.15) has a form similar to the standard KW duality equation, equation (2.10).

It is easy to prove that $d'(g^*) = \pm 1$. The maps we are looking for have $d'(g^*) = -1$, since the opposite sign leads to the trivial solution $d(g) \equiv g$. This is also shown in the appendix.

Consider now the symmetry properties of $\beta(g)$. We shall see that the KW duality symmetry property, equation (2.10), results in the beta-function being covariant under the operation $g \rightarrow d(g)$:

$$\beta(d(g)) = d'(g)\beta(g). \tag{2.16}$$

To prove it let us evaluate $\beta(d(g))$. Then equation (2.14) yields

$$\beta(d(g)) = \frac{2s(\tilde{g})(1 - s(\tilde{g}))}{(1 + s(\tilde{g}))(\mathrm{d}s(\tilde{g})/\mathrm{d}\tilde{g})}. \tag{2.17}$$

Bearing in mind equation (2.15), one is led to

$$\beta(d(g)) = \frac{2s(g) - 2}{s(g)(1 + s(g))(\mathrm{d}s(\tilde{g})/\mathrm{d}\tilde{g})}. \tag{2.18}$$

The derivative in the right-hand side of equation (2.18) should be rewritten in terms of $s(g)$ and $d(g)$. This may be easily done by applying equation (2.15):

$$\frac{\mathrm{d}s(\tilde{g})}{\mathrm{d}\tilde{g}} = \frac{\mathrm{d}}{\mathrm{d}\tilde{g}}\frac{1}{s(g)} = -\frac{s'(g)}{s^2(g)}\frac{1}{d'(g)}. \tag{2.19}$$

Substituting the right-hand side of equation (2.19) into (2.18) one obtains the desired symmetry relation, equation (2.16).

Therefore, the self-duality of the model allows us to determine the fixed-point value in another way, namely from the duality equation $d(g^*) = g^*$. One may now test the compatibility of this approach with the standard methods by computing $d(g)$ in the simplest approximation. For this purpose let us consider the function $s(g)$ given by equation (A.10) (in the appendix). Making use of a rough approximation, one obtains

$$s(g) \simeq \frac{2}{g} + \frac{24}{g^2} \simeq \frac{2}{g}\frac{1}{1 - 12/g} = \frac{2}{g - 12}. \tag{2.20}$$

Combining this Padé approximant with the definition of $d(g)$, equation (2.15), one is led to

$$d(g) = 4\frac{3g - 35}{g - 12}. \tag{2.21}$$

The fixed point of this function, $d(g^*) = g^*$, is easily seen to be $g^* = 14$. As expected, $d'(g^* = 14) = -1$.

Before moving to the next topic, we notice that the above-described approach may be regarded as another method for evaluating g^*, fully equivalent to the standard beta-function method. A systematic way of determining $d(g)$ has not yet been developed, in particular because of the specific analytical properties of $d(g)$. Namely, because $g = 0$ and ∞ are not regular points of $d(g)$, this function cannot be expanded in a Taylor series around these points.

3. Strong-coupling expansion for the beta-function

Our next purpose is to develop a strong-coupling expansion for the 2D scalar $g\Phi^4$ field theory. It has been mentioned already that nowadays the HT series expansions for $g(T > T_c, 0)$ are known rather well [13–20]. Equations (A.9) and (A.10), obtained in the appendix, form the basis of our treatment. These equations relate g to s, up to the ninth order in the appropriate expansion parameter. Having been equipped with these formulae, one may easily calculate the beta-function $\beta(g)$ as a power series in g^{-1}.

Inserting equation (A.10) into (2.14) and performing simple but somewhat cumbersome calculations, we are led to the desired asymptotic expansion for $\beta(g)$

$$\beta(g) = -2g + 32 - 64/g + 512/g^2 + 512/g^3 - 30\,720/g^4$$
$$-172\,032/g^5 + 32\,768/g^6 - 172\,032/g^7 + 32\,768/g^8 + O(g^{-9}). \tag{3.1}$$

From equation (3.1) it follows that in the large-g limit $\beta(g) \to -2g + 32$, whilst in the weak-coupling regime one has for $g \to 0$: $\beta(g) \to +2g$ [2–4, 23, 24]. It implies that the continuous function $\beta(g)$ changes sign at least once at some fixed point g^*.

Let us get some numerical estimates for g_+^* now, from equation (3.1), and compare these results with those found from the HT series expansions and those of the four-loop RG calculations. In the standard perturbative approach to quantum field theory at fixed dimension one must apply a resummation technique to the expansions of $\beta(g)$ and other RG functions. It is interesting that at least in low orders of perturbation theory the $1/g$ expansion (equation (3.1)) does not require the application of a resummation technique. The most reliable numerical estimates of g_+^* were obtained by means of the straightforward solution of the equation $\beta(g^*) = 0$, from equation (3.1), taken within the g^{-6} approximation (without the last two terms in g^{-7} and g^{-8}). The five (and best) subsequent approximation are as follows:

$$g_+^{*(1)} = 16 \qquad g_+^{*(2)} = 13.6568 \qquad g_+^{*(3)} = 15.0044$$
$$g_+^{*(4)} = 15.0784 \qquad g_+^{*(5)} = 14.7632. \tag{3.2}$$

Here the index k in $g_+^{*(k)}$ indicates that $k + 1$ terms are retained in the fixed-point equation under discussion. These estimates exhibit a regular behaviour, the last value being in very good agreement with the most recent estimate $g_+^* = 14.700 \pm 0.017$ obtained for the square lattice [14, 15]. The estimates obtained after taking into account the g^{-7} and g^{-8} terms differ significantly from the above values. This is apparently an indication that the $1/g$ series also require the application of some resummation technique.

Another approach to obtain a numerical estimate for g_+^* is a straightforward solution of equation (A.10), given in the appendix, after setting $s = 1$. In contrast to the fixed-point equation, $\beta(g) = 0$, it yields a rather poor value of the renormalized coupling constant, $g_+^* = 12.533$, compared to the value reported in [2–4].

It is interesting to compare our results with those obtained from the beta-function of the 2D Ising model and computed in the four-loop approximation, known to provide more or less satisfactory results for the critical indices [2, 3, 4]:

$$\beta(v) = 2v - 2v^2 + 1.432\,346v^3 - 1.861\,533v^4 + 3.164\,776\,4v^5 + O(v^6). \tag{3.3}$$

To obtain the beta-function in our normalization we have to change variables [24]

$$g = \tfrac{8}{3}\pi v \qquad \beta(g) = \tfrac{8}{3}\pi\beta(v). \tag{3.4}$$

The analysis based on the Padé–Borel method of resummation of asymptotic series yields $g_+^* = 15.08 \pm 2.5$ [24], which slightly exceeds the best values obtained from the HT series

calculations: $g_+^* = 14.70 \pm 0.017$ [14, 15]; $g_+^* = 14.67 \pm 0.04$ [24] (it is tempting to conjecture that $g_+^* = 14\pi/3$ [24]).

The exact numerical value of the critical exponent $\omega = \beta'(g^*)$ governing the leading corrections to the scaling laws [23, 25] is still unknown. The straightforward application of the above strong-coupling expansion results, equations (3.1) and (3.2), yields (within a conventional $\omega > 0$ definition) $\omega = \beta'(g^* = 14.76) = 1.88$. This estimate was obtained without exploiting a resummation procedure. It differs vastly from the estimate found from the four-loop RG calculations combined with the Padé–Borel method: $\omega = 1.3 \pm 0.2$ [24]. This value agrees well with that predicted by conformal field theory: $\omega = \frac{4}{3}$ [23].

4. Isothermal coupling constant and critical amplitudes

The two preceding sections were devoted to computing the approximate value of the renormalized coupling constant g_+^* at $h = 0$ in the isochore limit. Here we remark that in two dimensions there is a possibility of calculating the exact value of the renormalized coupling constant g_c^* in the isothermal limit, i.e. at the Curie point in an applied magnetic field, namely

$$g_c^* = \lim_{h \to 0} g(T = T_c, h) \tag{4.1}$$

by virtue of Cardy's formula [26]. It is, in fact, essential to stress that, in contrast to other isothermal critical amplitudes, g_c^* is fixed by this formula, which reads [26]

$$c = 3\pi h^2 (2 - \tfrac{1}{2}\eta)^2 \int d^2r \, r^2 G(r) \tag{4.2}$$

where c is a central charge; $G(r)$ is the two-spin correlation function and $\eta = 0.25$ is the anomalous scaling dimension of the spin variable σ in the 2D Ising model. Furthermore, equation (4.2) takes on a more convenient form

$$c = 3\pi (4 - \eta)^2 h^2 \xi^2 \chi \tag{4.3}$$

and it should be stressed that this formula is valid at $T = T_c$; $h \neq 0$. At the Curie point the correlation length and the susceptibility are described by power laws

$$\xi(h) = f_1^c h^{-2/(4-\eta)} \qquad \chi(h) = C_c h^{(2\eta-4)/(4-\eta)} \tag{4.4}$$

where f_1^c and C_c are the isothermal amplitudes. On the one hand, from equations (2.6) and (4.4) it follows that

$$g_c^* = \frac{2(2 - \eta)(4 - 3\eta)}{(4 - \eta)^2 C_c (f_1^c)^2}. \tag{4.5}$$

On the other hand, from equations (2.6) and (4.3) it is seen that the correlation length ξ drops out of the product $g_c^* c$:

$$c g_c^* = 3\pi (4 - \eta)^2 h^2 \left(-\frac{\partial^2 \chi / \partial h^2}{\chi} + 3\frac{(\partial \chi / \partial h)^2}{\chi^2} \right). \tag{4.6}$$

Inserting equation (4.4) into the right-hand side of equation (4.6), one obtains the renormalized coupling constant value at the end point of the isothermal line

$$g_c^* = \frac{6\pi}{c}(2 - \eta)(4 - 3\eta). \tag{4.7}$$

For the 2D Ising model equation (4.7) yields $g_c^* = 273\pi/4$. Equation (4.3) is easily seen to impose a constraint on the amplitudes f_1^c and C_c,

$$C_c(f_1^c)^2 = \frac{c}{3\pi(4-\eta)^2}. \tag{4.8}$$

From equation (4.8) it follows that what we actually found, by virtue of Cardy's formula, is only the product $C_c(f_1^c)^2$. To compute these quantities separately one needs more powerful techniques.

In some seminal papers [27–29] it was shown how to compute the isothermal amplitudes by making use of the thermodynamic Bethe ansatz and within the framework of the form-factor approach. In particular, in his paper Fateev obtained the following remarkable result [27]:

$$f_1^c = \frac{\Gamma(\frac{2}{3})\Gamma(\frac{8}{15})}{4\sin(\pi/5)\Gamma(\frac{1}{5})}\left[\frac{\Gamma(\frac{1}{4})\Gamma^2(\frac{3}{16})}{4\pi^2\Gamma(\frac{3}{4})\Gamma^2(\frac{13}{16})}\right]^{4/15} = 0.227\,019\,4675. \tag{4.9}$$

From equations (4.8) and (4.9) it follows that

$$C_c = 0.073\,199\,8414. \tag{4.10}$$

All of these results allow one to compute exactly the two following universal combinations [13] (see also [23, 30]):

$$Q_1 = \frac{C^c\delta}{(B^{\delta-1}C^+)^{1/\delta}}$$
$$Q_2 = \frac{C^+}{C^c}\left(\frac{f_1^c}{f_1^+}\right)^{2-\eta}. \tag{4.11}$$

We recall that the definitions of the critical amplitudes entering equation (4.11) are as follows [13, 23, 30]:

$$M_s = B(-\tau)^\beta$$
$$\chi(T \to T_c + 0) = C^+\tau^{-\gamma} \tag{4.12}$$
$$\xi = f_1^+\tau^{-\nu}$$

where M_s is the spontaneous magnetization. The amplitudes f_1^c and C_c have been defined in equation (4.4). The exact values are as follows: $\delta = 15$, $\eta = 0.25$, $\nu = 1$, $\beta = 0.125$, $\gamma = 1.75$, $B = 1.222\,410$ [13], $f_1^+ = 0.567\,296$ ([13], see also the appendix) and $C^+ = 0.962\,582$ [31]. Substituting all these values into equation (4.11) yields

$$Q_1 = 0.912\,648 \qquad Q_2 = 2.647\,714. \tag{4.13}$$

The exact values found provide a good opportunity to test the numerical results obtained from the HT series expansions and from Monte Carlo simulations. The fair estimates obtained from the analysis of HT series in the 2D Ising model on the square lattice yield $f_1^c = 0.233$ and $C_c = 0.0706$ [13], whilst the exact results are given by equations (4.9) and (4.10). As for the universal combinations $Q_{1,2}$, the series expansion analysis yields $Q_1 = 0.880\,23$, $Q_2 = 2.88$ [13].

Notice, in conclusion, that equations (4.7) and (4.8) also hold good for the general case of the 2D $g\Phi^4$ O(N)-symmetric model for $-2 < N < 2$, in particular for the minimal models of conformal field theory corresponding to the discrete values of N: $N = 2\cos(\pi/m)$; $m = 3, 4, \ldots, \infty$. More interesting still is that certain combinations of isothermal critical amplitudes, in particular $C_c(f_1^c)^2$, have the same numerical values both in the pure and in the quenched disordered 2D Ising model where $c = 0.5$; $\eta = 0.25$ [8].

5. Concluding remarks

We have proved the existence of the duality symmetry transformation $d(g)$ in the 2D $g\Phi^4$ theory such that $\beta(d(g)) = d'(g)\beta(g)$. Actually, this symmetry property was shown to result from the KW duality of the 2D lattice Ising model.

It would be tempting but wrong to regard $d(g)$ as a function connecting the weak- and strong-coupling regimes. For instance, this phenomenon takes place in the sinh Gordon theory with $\beta(d(g)) \equiv 0$, the exact S-matrix being invariant under the strong–weak-coupling duality $g \rightarrow 8\pi/g$ [28].

As a matter of fact, our proof is based on the properties of $g(s), s(g)$ defined only for $0 \leqslant s < \infty; g^* \leqslant g < \infty$ and therefore does not cover the weak-coupling region, $0 \leqslant g \leqslant g^*$. Whether the beta-function $\beta(g)$ does have the dual symmetry in the weak-coupling region, remains an open question.

In contrast to widely held views, the duality symmetry imposes only mild restrictions on $\beta(g)$. This means that this symmetry property fixes only even derivatives of the beta-function $\beta^{(2k)}(g^*)$ $(k = 0, 1, \ldots)$ at the fixed point, leaving the odd derivatives free. In fact, the duality equation $d(g) = g$ provides yet another method for determining the fixed point, independently of the approach based on the equation $\beta(g) = 0$. Another open problem is also that of finding a systematic approach for calculating $d(g)$.

In this paper the strong-coupling expansion for the beta-function of the 2D Ising model has been developed. This approach was shown to provide reliable results for the numerical values of g_+^*, which are found to be in good agreement with the best results from the HT series expansion. It is worth noting that g_+^* remains the same for the 2D Ising model with random bonds [8].

The method presented in this paper may be easily extended to the 2D O(N)-symmetric $g\Phi^4$ theories for arbitrary N. However, the above approach appears to be unable to provide reliable numerical results for the critical exponent ω, at least without the combined use of a resummation technique. In fact, the results obtained disagree both with estimates found from the four-loop RG calculations, combined with the Padé–Borel resummation method [24], and with the exact value $\omega = \frac{4}{3}$ given by conformal field theory [23]. As a matter of fact, the situation with the critical exponent ω looks somewhat unresolved. On the one hand, all the corrections to the scaling laws in the 2D Ising model are analytical. For instance, the susceptibility near T_c is given by $\chi = C^+\tau^{-7/4} + C_1^+\tau^{-3/4} + \cdots$ [31]. On the other hand, corrections to scaling are known to be powers of $\tau^{\omega\nu}$ [23]. All this would lead to $\omega = 1$, in obvious contradiction to conformal field theory. Moreover, the spectrum of conformal dimensions of the 2D Ising model consists of just three numbers, these being $(0, \frac{1}{8}, 1)$. That is the reason why the appearance of an operator with a fractional scaling dimension $\frac{4}{3}$ is so far unclear [23]. Thus, although one can make a meaningful comparison between the numerical values obtained, nevertheless the apparent spread of these numerical results for ω does not allow one to construct a smooth interpolation for $\beta(g)$ between the two regions: $g < g_+^*$ and $g > g_+^*$.

Regarding critical amplitudes, namely, f_1^c, C_c, Q_1, Q_2, we can conclude that the known numerical results are in a surprisingly good agreement with the exact results. For further discussion, see [32].

We have already seen that the exact beta-function $\beta_{\text{Ising}}(T)$ is essentially non-perturbative. Non-perturbative terms give rise to non-trivial fixed points in contrast to the standard beta-functions of nonlinear sigma models with continuous symmetry, in particular for the O(N)-symmetric theory with $N > 2$ [33]. The very existence of non-perturbative terms is indicative of the mathematical illegitimacy of the naive analytical continuation from the

$N > 2$ to the $N < 2$ region often employed in statistical mechanics and condensed matter physics.

Appendix

Here we begin with giving a simple derivation of the correlation length ξ for the 2D Ising model from the high-temperature expansions and will then present some useful relations.

The HT expansions up to $K^8 (K \equiv J/T)$ are as follows [15]:

$$\chi = 1 + 4K + 12K^2 + 104K^3/3 + 92K^4 + 3608K^5/15$$
$$+ 3056K^6/5 + 484\,528K^7/315 + 400\,012K^8/105 \tag{A.1}$$

$$\mu_2 = 4K + 32K^2 + 488K^3/3 + 2048K^4/3 + 38\,168K^5/15$$
$$+ 394\,624K^6/45 + 8\,994\,736K^7/315 + 28\,064\,768K^8/315 \tag{A.2}$$

$$\chi_{hh}'' = -2 - 32K - 264K^2 - 4864K^3/3 - 8232K^4 - 553\,024K^5/15$$
$$- 2\,259\,616K^6/15 - 180\,969\,728K^7/315 - 217\,858\,792K^8/105 \tag{A.3}$$

where χ_{hh}'' is the second derivative of the homogenous susceptibility with respect to a magnetic field h; χ and μ_2 were defined in section 2.

The standard RG equation for the effective temperature T is given by

$$\xi \frac{dT}{d\xi} = \beta_{\text{Ising}}(T) \tag{A.4}$$

where ξ is the correlation length (equation (2.3))

$$\xi^2 = \frac{\mu_2}{2d\chi}. \tag{A.5}$$

The key observation for computing ξ is to make use of the new variable $s = \exp(2K) \tanh(K)$. One has to substitute equations (A.1) and (A.2) into equation (A.5) and then to rewrite the expression obtained in terms of s. At this order of approximation the procedure gives the result

$$\xi^2 = s + 2s^2 + 3s^3 + 4s^4 + 5s^5 + 6s^6 + 7s^7 + 8s^8 + O(s^9)$$
$$= \frac{s}{(1-s)^2}. \tag{A.6}$$

We believe the closed-form above to be an exact result (it satisfies in particular the $s \to s^{-1}$ duality symmetry). However, it should not be confused with other exact formulae available in the literature for ξ, based for instance on the transfer-matrix eigenvalue approach [34] rather than our field-theoretic definition.

One sees that the correlation length as given by equation (A.6) in its closed form does exhibit the correct behaviour near the Curie point,

$$\xi = f_1^+ \frac{1}{\tau} \qquad \tau = \frac{T - T_c}{T_c} \qquad T_c = \frac{2}{\ln(\sqrt{2} + 1)} = 2.269\,185$$

$$f_1^+ = \frac{1}{4\ln(\sqrt{2} + 1)} = 0.567\,296 \tag{A.7}$$

where f_1^+ is the correlation length critical amplitude [13].

Combining equations (A.4) and (A.6) and carrying out some calculations, we arrive at a nice expression for the beta-function written in terms of s,

$$\frac{\mathrm{d}s}{\mathrm{d}\ln\xi} = \beta_{\text{Ising}}(s) = \frac{2s(1 - s)}{1 + s}. \tag{A.8}$$

Now substituting equations (A.1)–(A.3) into the right-hand side of equation (2.9) and making use of the variable s, one is led to some helpful relations linking s and g,

$$\frac{1}{g} = s/2 - 3s^2 + 39s^3/2 - 138s^4 + 2061s^5/2$$

$$-7909s^6 + 122\,907s^7/2 - 480\,012s^8/945 + O(s^{-9}) \tag{A.9}$$

$$s = 2/g + 24/g^2 + 264/g^3 + 2976/g^4 + 35136/g^5 + 423\,680/g^6$$

$$+5\,149\,824/g^7 + 63\,275\,520/g^8 + O(g^{-9}). \tag{A.10}$$

Rewritten in terms of T, equation (A.8) reads

$$\frac{\mathrm{d}T}{\mathrm{d}\ln\xi} = \beta_{\text{Ising}}(T) = T^2 \frac{1 - 2\exp(-2/T) - 2\exp(-4/T) + 2\exp(-6/T) + \exp(-8/T)}{1 + 2\exp(-2/T) + 2\exp(-6/T) - \exp(-8/T)}. \tag{A.11}$$

The exact beta-function given by equations (A.8) and (A.11) exhibits the following properties:

(a) it vanishes just at two fixed points (FP): at the Gaussian FP $s = 0$ and at the Ising FP $s = 1$;

(b) the derivative of the beta-function at the Ising FP gives the critical exponent of the correlation length ξ, more precisely

$$\nu^{-1} = -\frac{\mathrm{d}\beta_{\text{Ising}}(s)}{\mathrm{d}s}\bigg|_{s=1} = 1 \tag{A.12}$$

(c) the beta-function is covariant under the duality transformation, namely

$$\beta_{\text{Ising}}(s) = -s^2 \beta_{\text{Ising}}\left(\frac{1}{s}\right) \tag{A.13}$$

and this equation has been referred to as the consistency condition for the beta-function [11];

(d) these properties agree fully with results from the HT series expansions.

The duality property (c) implies that the RG equation under discussion takes the same form both in the dual and in the original variables. We note in passing that the well known self-dual beta-function obtained within the Migdal approximation for $d = 2$, which reads [35]

$$\beta_{\text{Migdal}}(T) = -T - \frac{T^2}{2}\sinh\frac{2}{T}\ln\tanh\frac{1}{T} \tag{A.14}$$

does not satisfy either properties (b) and (d) [35].

To end with, we prove some useful relations concerning the duality transformation $\tilde{g} = d(g)$ introduced in section 2.

(a) Let us show that $d'(g^*) = \pm 1$. First of all, from the definition

$$d(g) \equiv s^{-1}\left(\frac{1}{s(g)}\right) \tag{A.15}$$

we have that $d(d(g)) = g$. Differentiating this with respect to g one obtains

$$\begin{aligned}
d(d(g)) &= g & d(g^*) &= g^* \\
d'(d(g))\,d'(g) &= 1 & g &= g^* \\
d'(g^*)\,d'(g^*) &= 1
\end{aligned} \tag{A.16}$$

showing that indeed $d'(g^*) = \pm 1$.

(b) Differentiating equation (A.16) (second from top) with respect to g one obtains

$$d'[d(g)]d'(g) = 1 \qquad d''[d(g)]d'(g)^2 + d'[d(g)]d''(g) = 0 \tag{A.17}$$

and at the fixed point we arrive at $(d(g^*) = g^*)$:

$$d''(g^*)(d(g^*))^2 + d'(g^*)d''(g^*) = 0. \tag{A.18}$$

If $d'(g^*) = -1$ we have an identity $d''(g^*) = d''(g^*)$. In the opposite case from $d'(g^*) = +1$ it follows that $d''(g^*) = 0$. Proceeding in the same way, it is easy to see that all higher derivatives vanish identically at the fixed point. This, alone, does not imply that $d(g) \equiv g$, since it could be that $d(g) = g + f(g)$ where $f(g)$ is some non-analytic function having vanishing derivatives at g^*. Let us then assume that $f(g)$ is the first term of an asymptotic expansion of $d(g)$ around the fixed point, so that $f(g) \to 0$ as $g \to g^*$. Then remembering that $d(d(g)) = g$,

$$g = d(d(g)) = d(g + f(g)) = g + f(g) + f(g + f(g)) \simeq g + 2f(g) \tag{A.19}$$

and we do obtain $f(g) \equiv 0$.

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